АРИФМЕТИЧЕСКИЕ СВОЙСТВА РЕШЕТОК $\omega$-ВЕЕРНЫХ ФОРМАЦИЙ КОНЕЧНЫХ ГРУПП

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Abstract. Only finite groups and classes of finite groups are considered. The lattice approach to the study of formations of groups was first applied by A.N. Skiba in 1986. L.A. Shemetkov and A.N. Skiba established main properties of the lattices of local formations and $\omega$ -local formations where $\omega$ is a nonempty subset of the set $\mathcal{P}$ of all primes. An $\omega$ -local formation is one of types of $\omega$ -fibered formations introduced by V.A. Vedernikov and M.M. Sorokina in 1999. Let a function $f : \omega \cup \{ \omega' \} \rightarrow \{ \text{formations of groups} \}$ be an $\omega F$ -function; the function $\delta : \mathcal{P} \rightarrow \{ \text{nonempty Fitting formations} \}$ be a $\mathcal{P} FR$ -function. A formation $\omega F(f, \delta) = (G : G / O_\omega(G) \in f(\omega))$ with $G / G_{\delta(p)} \in f(p)$ for all $p \in \omega \cap \pi(G)$, where $O_\omega(G)$ and $G_{\delta(p)}$ are the largest normal $\omega$ -subgroup of the group $G$ and the $\delta(p)$ -radical of $G$ respectively, is called an $\omega$ -fibered formation with the direction $\delta$ and with the $\omega$ -satellite $f$. We study properties of the lattices of $\omega$ -fibered formations of groups with the given direction $\delta$. We have established arithmetic properties of the lattice $\Theta_{\omega\delta}(F)$ of all $\omega$ -fibered subformation with the direction $\delta$ of the formation $F$.

Аннотация. Рассматриваются только конечные группы. Решеточный подход к изучению формаций групп был впервые изложен А.Н. Скибой в 1986 году. Л.А. Шеметковым и А.Н. Скибой были установлены основные свойства решеток локальных и $\omega$ -локальных формаций, где $\omega$ – непустое подмножество множества $\mathcal{P}$ всех простых чисел. $\omega$ -Локальные формации являются одним из видов $\omega$ -веерных формаций, введенных в рассмотрение В.А. Ведерниковым и М.М. Сорокиной в 1999 году. Функция $f : \omega \cup \{ \omega' \} \rightarrow \{ \text{формации групп} \}$ называется $\omega F$ -функцией; функция $\delta : \mathcal{P} \rightarrow \{ \text{непустые формации Фиттинга} \}$ называется $\mathcal{P} FR$ -функцией. Формация $\omega F(f, \delta) = (G : G / O_\omega(G) \in f(\omega))$ с $G / G_{\delta(p)} \in f(p)$ для всех $p \in \omega \cap \pi(G)$, где $O_\omega(G)$ и $G_{\delta(p)}$ – наибольшая нормальная $\omega$ -подгруппа группы $G$ и $\delta(p)$ -радикал группы $G$ соответственно, называется $\omega$ -веерной формацией с направлением $\delta$ и $\omega$ -спутником $f$. В данной статье изучаются решетки $\omega$ -веерных формаций с заданным направлением $\delta$. Авторами установлены арифметические свойства решетки $\Theta_{\omega\delta}(F)$ всех $\omega$ -веерных подформаций с направлением $\delta$ формации $F$.

Key words: finite group, class of groups, formation, $\omega$ -fibered formation, lattice of formations.

Ключевые слова: конечная группа, формация, $\omega$ -веерная формация, решетка формаций.

Introduction

Formations and Fitting classes of finite groups plays an important role in the theory of the classes of finite groups. Local and composition formations and Fitting classes of finite groups are the well-known types of the classes of finite groups. Local formations were constructed by W. Gaschütz [3], L.A. Shemetkov in [9] introduced a concept of a composition formation, B. Hartley in [4] defined a local Fitting class. The most important results about local and composition formations and local Fitting classes of finite groups have been presented in [2, 8, 11]. Later L.A. Shemetkov and A.N. Skiba constructed $\omega$ -local formations and Fitting classes [12] and $L$ -composition formations [13]. Local and composition formations and Fitting classes ($\omega$ -local and $\Omega$ -composition formations...
and Fitting classes) of finite groups are constructed using the function named a-satellite.

In 1999 V.A. Vedernikov introduced \( \omega \)-fibered formations and Fitting classes (see, for instance, [16, 17]) and \( \Omega \)-foliated formations and Fitting classes of the finite groups. Note that \( \omega \)-fibered and \( \Omega \)-foliated formations and Fitting classes are the generalizations of \( \omega \)-local and \( \Omega \)-composition formations and Fitting classes respectively. Formations and Fitting classes constructed by V.A. Vedernikov are defined via two functions: a function-satellite that have mentioned above and a new function named a direction of the formation (of the Fitting class). There exists infinite number of different functions-directions. Using the different types of the functions-directions many new types of formations and Fitting classes of finite groups have been discovered (see, for instance, [18]) and have been studied (see, for instance, [6]). Notice that an \( \omega \)-local formation (an \( \omega \)-local Fitting class) is a representative of the series of \( \omega \)-fibered formations (\( \omega \)-fibered Fitting classes) and an \( L \)-composition formation is a representative of the series of \( \Omega \)-foliated formations.

The lattice methods play an important role in the study of the properties of many algebraic objects. A partially ordered set is called a lattice if for any two elements there exist an exact lower bound (the lattice intersection) and an exact upper bound (the lattice union). The lattice approach to the study of formations of groups was first applied by A.N. Skiba in 1986 (see, for instance, [14]). L.A. Shemetkov and A.N. Skiba established main properties of lattices of local formations and \( \omega \)-local formations (see, for instance, [10, 12]).

Research goals and objectives. The main goal of the research is to study new properties of the lattices of \( \omega \)-fibered formations of groups with the given direction \( \delta \). We solved the following tasks:

- we established properties of the lattice \( \Theta_{\omega \delta}(F) \) of all \( \omega \)-fibered subformation with the direction \( \delta \) of the formation \( F \) in the case \( |\Theta_{\omega \delta}(F)| = 2 \) (Theorem 1) and in the case \( |\Theta_{\omega \delta}(F)| = 3 \) (Theorem 3).

- we established conditions under which the lattice \( \Theta_{\omega \delta}(F) \) of all \( \omega \)-fibered subformations of the given \( \omega \)-fibered formation \( F \) with the direction \( \delta \) has a finite number of elements not more 3 (Theorem 2).

Research methods. The authors use methods of proofs of the abstract theory of finite groups, as well as methods of the theory of classes of groups, in particular, methods of the theory of formations.

Research novelty. All the results obtained are new and can be used in the further study of lattice properties of formations of finite groups.

Preliminary information

Only finite groups are considered. The symbol := means the equality by the definition. Notations and definitions of groups and classes of groups are standard (see, for instance, [2]). We give just some of them. A subgroup \( N \) of the group \( G \) is called a normal subgroup of the group \( G \) and denoted by \( N \triangleleft G \) if \( Ng = gN \) for any \( g \in G \). A group \( G \) is called monolithic if \( G \) has a unique minimal normal subgroup (\( a \) monolith). Recall that a class of groups is a set of groups such that with every its group \( G \) it contains all groups isomorphic to the group \( G \). By \( F_{1}F_{2} \) we denote the product of classes \( F_{1} \) and \( F_{2} \), i.e. \( F_{1}F_{2} = \{ g \in E \} \) there exists \( N \triangleleft G \) such that \( N \in F_{1} \), \( G/N \in F_{2} \) [2]. A class of groups \( F \) is called a formation if \( F \) satisfies two following conditions:

1) \( G \in F \) and \( N \triangleleft G \) imply that \( G/N \in F \) (in other words, \( F \) is \( Q \)-closed);

2) \( G/N_{1} \in F \) and \( G/N_{2} \in F \) imply that \( G/(N_{1} \cap N_{2}) \in F \) (\( F \) is \( R_{0} \)-closed).

A class of groups \( H \) is called a Fitting class if \( H \) satisfies two following conditions:

1) \( G \in H \) and \( N \triangleleft G \) imply that \( N \in H \) (in other words, \( H \) is \( S_{\omega} \)-closed);

2) \( G = N_{1}N_{2}, N_{1} \triangleleft G, N_{2} \triangleleft G, N_{1}, N_{2} \in H \) imply that \( G \in H \) (\( H \) is \( R \)-closed).

Let \( F \) and \( H \) be a formation and a Fitting class respectively. The smallest normal subgroup \( N \) of the group \( G \) such that \( G/N \in F \) is called an \( F \)-coradical of the group \( G \) and it is denoted by \( G^{F} \); the largest normal subgroup \( N \) of the group \( G \) such that \( N \in H \) is called an \( H \)-radical of the group \( G \) and it is denoted by \( G_{H} \).

Let \( \mathcal{F} \) be a set of all primes, \( \omega \) be a nonempty subset of \( \mathcal{F} \). By \( \pi(G) \) we denote the set of all prime dividers of the order of the group \( G \). A group \( G \) is called an \( \omega \)-group if \( \pi(G) \subseteq \omega \); a group \( G \) is called an \( \omega' \)-group if \( \pi(G) \cap \omega = \emptyset \). A group \( G \) is called an \( \omega \)-solvable group if every chief factor of the group \( G \) is an \( \omega' \)-group or abelian \( p \)-group for some \( p \in \omega \). A group \( G \) is called an \( \omega \)-separable group if for every chief factor \( A/B \) of the group \( G \) it is true that \( |\pi(A/B) \cap \omega| \leq 1 \).

By (1) it is denoted the class of all identity groups; by \( E \) it is denoted the class of all finite groups; by \( E_{\omega} \) it is denoted the class of all \( \omega \)-groups; by \( N \) it is denoted the class of all nilpotent groups. Let \( p \in \mathcal{F} \). By \( Z_{p} \) it is denoted the group of the order \( p \); \( N_{p} \) is the class of all \( p \)-groups; \( E_{p} \) is the class of all \( p' \)-groups; \( E_{(2p)} \) is the class of all such groups that don't have composition factors which are isomorphic to \( Z_{2p} \); \( S_{2p} \) is the class of all groups whose every chief \( p \)-factor is central. The class generated by the set \( X \) of groups is denoted by \( \langle X \rangle \), i.e. \( \langle X \rangle \) is an intersection of all classes of groups containing \( X \); in particular, \( \langle G \rangle \) is the class of all groups which are isomorphic to the group \( G \); (1) is the class of all identity groups. The formation generated by the set \( X \) of groups is denoted by \( \mathcal{F}(X) \), i.e. \( \mathcal{F}(X) \) is an intersection of all formations containing \( X \). For the class \( X \) we put \( \mathcal{F}(X) = \bigcup_{G \in X} \pi(G) \). Further, \( \omega \) is a nonempty subset of the set \( \mathcal{F} \). Let \( f : \omega \cup \{ \omega' \} \rightarrow \{ \text{formations of groups} \} \) be an \( \omega \)-function. Additionally, let \( \omega \). A formation \( \mathcal{F}(X, \delta) = \langle G: G/\Omega_{\delta}(G) \in f(\omega') \rangle \) with \( \Omega_{\delta}(G) \) and \( \Omega_{\delta}(p) \) are the largest normal \( \omega \)-subgroup of the group \( G \) and the \( \delta(p) \)-radical of \( G \) respectively, is called an \( \omega \)-fibered formation with the direction \( \delta \) and with the \( \omega \)-satellite \( f \) (briefly an \( \omega \)-fibered formation). An \( \omega \)-satellite \( f \) of the \( \omega \delta \)-fibered formation \( F \) is called inner if \( f(\omega) \subseteq F \) for any \( x \in \omega \cup \{ \omega' \} \). By \( \mathcal{F}(X, \delta) \)
it is denoted an \( \omega \delta \)-fibered formation generated by the set \( X \) of groups, i.e. \( \omega F(X, \delta) \) is an intersection of all \( \omega \delta \)-fibered formations containing \( X \). Let \( f_1 \) and \( f_2 \) be \( \omega F \)-functions (\( \mathbb{F} \text{-functions} \)). We let \( f_1 \leq f_2 \) if \( f_1(x) \subseteq f_2(x) \) for every \( x \in \omega \cup \{\omega\} \) (for every \( x \in \omega \)); we put \( f_1 < f_2 \) if \( f_1 \leq f_2 \) and \( f_1 \neq f_2 \). An \( \omega \delta \)-fibered formation with the direction \( \delta \) is called: \( \omega \)-absolute if \( \delta = \delta_0 \) where \( \delta_0(p) = E_p \) for any \( p \in \omega \); \( \omega \)-local if \( \delta = \delta_1 \) where \( \delta_1(p) = E_p N_p \) for any \( p \in \omega \); \( \omega \)-special if \( \delta = \delta_2 \) where \( \delta_2(p) = E_p \) \( N_p \) for any \( p \in \omega \); \( \omega \)-central if \( \delta = \delta_3 \) where \( \delta_3(p) = S_{cp} \) for any \( p \in \omega \).

It follows directly from these definitions that \( \delta_0 < \delta_1 < \delta_2 < \delta_3 \). The direction \( \delta \) of the \( \omega \delta \)-fibered formation is called a \( bc \)-direction if \( \delta \) is a \( bc \)-direction, i.e. \( \delta(p) N_p = \delta(p) \) for any \( p \in \omega \), and \( \delta \) is a \( p \)-direction, i.e. \( E_p \delta(p) = \delta(p) \) for any \( p \in \omega \) [16].

A partially ordered set \( \Theta \) is called a lattice if for any two elements there exist an exact lower bound (the lattice intersection) and an exact upper bound (the lattice union). In the lattice \( \Theta \) the lattice intersection of the elements \( x \) and \( y \) is denoted by \( x \land \_y \) and the lattice union of the elements \( x \) and \( y \) is denoted by \( x \lor \_y \) [1]. Let \( \Theta \) be a nonempty set of formations which is partially ordered regarding the inclusion \( \subseteq \), \( F_1 \) and \( F_2 \) are \( \Theta \)-formations (i.e. \( F_1, F_2 \in \Theta \)). Then the lattice intersection and the lattice union of the formations \( F_1 \) and \( F_2 \) are defined respectively as \( F_1 \land \_F_2 = F_1 \cap F_2 \) (a), \( F_1 \lor \_F_2 = \Theta \), where \( \Theta \) is a \( \Theta \)-formation generated by the union \( F_1 \cup F_2 \). \( \Theta \) is an intersection of all \( \Theta \)-formations containing \( F_1 \cup F_2 \). The set \( \Theta \) of formations is called a complete lattice of formations if the intersection of any set of \( \Theta \)-formations is a \( \Theta \)-formation and there exists a formation \( M \) in \( \Theta \) such that \( F \subseteq M \) for any formation \( F \) in \( \Theta \). For a given lattice \( \Theta \) of formations by \( \Theta \) we denote the set of all \( \Theta \)-subformations of the formation \( F \) [11].

Let \( \delta \) be an arbitrary \( \mathbb{F} \text{-function} \). Denote the set of all \( \omega \delta \)-fibered formations of finite groups by \( \Theta_{\omega \delta} \). By \( \Theta_{\omega}(F) \) we denote the set of all \( \omega \delta \)-fibered subformations of the \( \omega \delta \)-fibered formation \( F \). Denote the set of all formations of finite groups by \( \Theta_{\omega} \).

The following results are well-known.

**Lemma 1 (Theorem 5, [16, p. 49]).** Let \( X \) be a non-empty class of groups. Then the \( \omega \delta \)-fibered formation \( F = \omega F(X, \delta) \) with the direction \( \delta \) where \( \delta_0 \leq \delta \) has an unique minimal \( \omega \delta \)-satellite \( f \) such that:

\[
\omega(F) = form(G/\omega F(G))(G \in X). \\
(\omega)(p) = form(G/\omega F(G))(G \in X) \text{ for any } p \in \omega \cap \pi(X). \\
\omega(F) = \emptyset \text{ for any } q \in \omega \backslash \pi(X).
\]

**Lemma 2 (Corollary 5.1, [16, p. 49]).** Let \( f_1 \) be a minimal \( \omega \delta \)-satellite of the \( \omega \delta \)-fibered formation \( F_1 \) with the direction \( \delta \) where \( \delta_0 \leq \delta \). Then \( F_1 \subseteq F_2 \) if and only if \( f_1 \leq f_2 \).

**Lemma 3 (Corollary 4.4, [15, p. 53]).** Let \( (1) \neq F \subseteq E \). Then \( \Theta_{\omega}(F) \) is \{(1), F\} and only if then \( F = \omega F(A) \) where \( A \) is a simple group.

**Remark 1.** Let \( p \in \omega \). Then the class \( N_p \) is an \( \omega \delta \)-fibered formation with the \( \omega \delta \)-satellite \( f \) and the direction \( \delta \) where \( \delta \) is a \( b \)-direction and \( f \) is an \( \omega F \)-function which has the following structure:

\[
f(\omega) = N_p, \quad f(p) = \{1\} \text{ and } f(q) = \emptyset \text{ for any } q \in \omega \backslash \{p\}.
\]

**Remark 2.** The class \( (1) \) of all identity groups is an \( \omega \delta \)-fibered formation with the \( \omega \delta \)-satellite \( f \) and the direction \( \delta \) where \( \delta \) is an arbitrary \( \mathbb{F} \text{-function} \) and \( f \) is an \( \omega F \)-function which has the following structure:

\[
f(\omega) = \emptyset, \quad f(p) = \emptyset \text{ for any } q \in \omega \backslash \{p\}. \text{ Then } F = N_p.
\]

**Lemma 4.** Let \( F = \omega F(f, \delta) \) where \( \delta \) is a \( b \)-direction, \( \delta_0 \leq \delta \), \( p \in \omega \), \( f \) is an \( \omega F \)-function such that \( f(\omega) = (1), f(p) = \emptyset \text{ for any } q \in \omega \backslash \{p\} \). Then \( F = N_p \).

**Proof.** According to Remark 1 the class \( N_p \) is an \( \omega \delta \)-fibered formation with the direction \( \delta \). Since \( \delta_0 \leq \delta \) and \( N_p = \omega F(N_p, \delta) \) then, according to Lemma 1, the formation \( N_p \) has a unique minimal \( \omega \delta \)-satellite \( h \) which has the following structure:

\[
h(\omega) = form(\delta(\omega) (G \in N_p)), \\
h(p) = form(G/\gamma(G)(G \in N_p), h(q) = \emptyset \text{ for any } q \in \omega \backslash \{p\}.
\]

Establish that \( F = N_p \). Since \( N_p = \omega F(h, \delta) \) and \( F = \omega F(f, \delta) \) then, according to the definition of the \( \omega \delta \)-fibered formation, it is sufficient to verify that \( f = h \). Note that \( h(q) = \emptyset = f(q) \) for any \( q \in \omega \backslash \{p\} \).

1. Show that \( h(\omega) = f(\omega) \). Let \( G \in N_p \). Since \( p \in \omega \) then \( O_p(G) = \gamma_p(G) = G \). Thus, \( G/\gamma(G) = (1) \) and, consequently, for any group \( G \in N_p \) it is true that \( G/O_p(G) = (1) \). Then \( h(\omega) = form(1) = (1) = f(\omega) \).

2. Proof that \( h(p) = f(p) \). Let \( G \in N_p \). Since \( \delta \) is a \( b \)-direction then \( \delta(p) N_p = \delta(p) \) and, consequently, \( G \in \delta(p) \). Thus, \( G/G_\delta(p) = G \). Hence, for any group \( G \in N_p \) it is true that \( G/G_\delta(p) = 1 \). Then \( h(p) = (1) = f(p) \).

From clauses 1 and 2 we obtained that \( f = h \) and, so, \( \omega F(h, \delta) = \omega F(f, \delta) \). Hence, \( F = N_p \). The lemma is proved.

**Main results**

**Theorem 1.** Let \( \delta = \omega F \)-function, \( \delta_0 \leq \delta \), \( F, \omega F, \delta_0 \subseteq \omega \). If \( |\Theta_{\omega}(F)| = 2 \), then \( F = \omega F(A, \delta) \) where \( A \) is a simple group.

**Proof.** Let \( |\Theta_{\omega}(F)| = 2 \). Then, in view of the condition of the theorem and of Remark 2, \( \Theta_{\omega}(F) = \{(1), F\} \). Establish that \( F = \omega F(A, \delta) \) where \( A \) is a simple group. Since \( F \neq (1) \) there exists a non-identity group \( G \in F \). From \( (1) \neq \omega F(G, \delta) \subseteq F \) it follows that \( \omega F(G, \delta) = F \). If \( G \) is a simple group then the theorem is proved. Suppose that \( G \) is not a simple group. Repeating similar reasoning for the
group the $G_1$ we may obtain that $F = \omega F(G_1/N_\omega \delta)$ where $N_1 \unlhd G_1$ and $1 \neq N_1 \neq G_1$. If $G_1/N_1 = G_2$ is a simple group then the theorem is proved. Suppose that $G_2$ is not a simple group. Since $G$ is a finite group then repeating similar reasonings in the finite number of steps we may obtain that $F = \omega F(G_m/N_m \omega \delta)$ where $A_1 = G_m/N_m$ is a simple group.

Thus, $F = \omega A(\delta)$ where $A$ is a simple group. The theorem is proved.

**Theorem 2:** Let $F = \omega F(A, \delta)$ where $A$ is a simple $\omega$-separable group, $\delta$ is a $b$-direction, $\delta_0 \leq \delta$. Then $|\theta_{\omega A}(F)| \leq 3$.

**Proof:** Since $(1) \in \theta_{\omega A}(F)$, $\delta \in \theta_{\omega A}(F)$ and $F \neq (1)$ then $|\theta_{\omega A}(F)| \geq 2$. According to the definition of the $\omega \delta$ -fibered fibration, it is true that $\emptyset \not\in \theta_{\omega A}$ and, so, $\emptyset \not\in \theta_{\omega A}(F)$. Let $H \in \theta_{\omega A}(F)$, $H \not< F$, and suppose that $h$ and $f$ the minimal $\omega$ -satellites of the $\omega \delta$ -fibered formations $H$ and $F$ respectively. Based on Lemma 2 $h \not< f$ and in a view of Lemma 1 $h$ and $f$ has the following structure:

$h(\omega) = \omega F(G_1/\omega \delta(G)|G \in H)$

$h(p) = \omega A(G_1/\omega \delta\omega(A)|G \in H)$

for any $p \in \omega \cap \pi(H)$.

$h(q) = \emptyset$ for any $q \in \omega \\cap \pi(H)$.

$f(\omega) = \omega A(G_1/\omega \delta\omega(A)|\in H)$

$f(p) = \omega F(A/\delta\omega(A)|p \in \omega \cap \pi(A)$.

$f(q) = \emptyset$ for any $q \in \omega \cap \pi(A)$.

It follows from the definition of the $\omega F$ -function that $f(\omega) \not= \emptyset$ and $h(\omega) \not= \emptyset$. Since $h \not< f$ then $h(q) = f(q) = \emptyset$ for any $q \in \omega \cap \pi(A)$. According to Remark 2, it is true that $(1) = \omega A(m\delta)$ where $m(\omega) = (1)$ and $m(p) = \emptyset$ for any $p \in \omega$.

Since $A$ is a simple $\omega$ -separable group and $A/\pi(A) = 1$ is a unique chief factor of the group $A$, then $|\omega \cap \pi(A)| \leq 1$. It means that the group $A$ is an $\omega'$ -group or $\omega \cap \pi(A) = \{p\}$ for some prime $p$.

1. Let $A$ be an abelian $p$-group. Since $p \in \omega$ then $\omega A(A) = A$ and, so, $h(\omega) = f(\omega) = (1)$. Consider the formation $f(p)$. In view of the fact that $\delta$ is a $b$-direction, it is true that $A \in \delta(p)$ and, consequently, $A_{\delta(p)} = A$. Thus, $A/A_{\delta(p)} = 1$ and $f(p) = (1)$. Since $h \not< f$ it follows $h(p) = \emptyset$. It means that $H = (1)$.

2. Let $A$ be a non-abelian $p$-group. Since $\omega \cap \pi(A) = \{p\}$ then $\omega A(\omega p) = 1$ and $f(\omega) = \omega A(\omega p)$. If $h(\omega) = f(\omega)$ then $A \in \omega A(\omega) \subseteq H$ and $F = \omega F(A, \omega) \subseteq H$ which is a contradiction. Consequently, $h(\omega) \not= f(\omega)$.

According to Lemma 3, $h(\omega) = (1)$. Consider the formation $f(p)$. Since $A$ is a simple group and $A_{\delta(p)} \not< A$ then $A/A_{\delta(p)} = A$ or $A_{\delta(p)} = 1$. Let $A/A_{\delta(p)} = A$.

Then $f(p) = (1)$. There exist two cases for the formation $h(p)$: $h(p) = \emptyset$ or $h(p) = (1)$. If $h(p) = \emptyset$ then $h = m$ and $H = (1)$. If $h(p) = (1)$ then, according to Lemma 4, $H = N_p$.

Let $A_{\delta(p)} = 1$. Then $h(p) = \omega F(A)(1)$. Since $A \not< H$ then $h(p) \in f(p)$. It means that $h(p) = \emptyset$ or $h(p) = (1)$. If $h(p) = \emptyset$ then $H = (1)$. If $h(p) = (1)$ then, according to Lemma 4 $H = N_p$.

Thus, we obtained that $H = (1)$ and, consequently, in this case $|\theta_{\omega A}(F)| = 2$, or $H = N_p$ and, so, in this case $|\theta_{\omega A}(F)| = 3$. It means that $|\theta_{\omega A}(F)| \leq 3$. The theorem is proved.

**Corollary 1:** Let $F = \omega F(A, \delta)$ where $A$ is a simple $\omega$-solvable group, $\delta$ is a $b$-direction, $\delta_0 \leq \delta$. Then $|\theta_{\omega A}(F)| = 2$.

**Proof:** In view of the fact that $A$ is a simple $\omega$-solvable group then $A$ is a simple $\omega$ -separable group. According to the Theorem 2, $|\theta_{\omega A}(F)| \leq 3$. Since the case 2.2 from the proof of the Theorem 2 is impossible for the $\omega$-solvable group it follows that $\theta_{\omega A}(F) = \{(1)\}$. Thus, $|\theta_{\omega A}(F)| = 2$. The corollary is proved.

**Theorem 2:** Let $F = \omega F(A, \delta)$ where $A$ is a simple $\omega'$ -group, $\delta$ is a $b$-direction, $\delta_0 \leq \delta$. Then $|\theta_{\omega A}(F)| = 2$.

**Proof:** Since $A$ is a simple $\omega'$ -group then there will be valid only the case 1 from the proof of the Theorem 2. Thus, according to the proof of the Theorem 2, it follows that $\theta_{\omega A}(F) = \{(1)\}$. Consequently, $|\theta_{\omega A}(F)| = 2$. The corollary is proved.

**Theorem 3:** Let $\delta$ be a $b$-direction, $\delta_0 \leq \delta$, $F \in \theta_{\omega A}$ and $F$ be an $\omega$-solvable formation. If $|\theta_{\omega A}(F)| = 3$ then $F = \omega F(A, \delta)$ where $A$ is a monolithic group with a monolith $R$ and $\omega F(G/R, \delta)$ is the unique maximal $\omega \delta$-fibered subformation of $F$.

**Proof:** Let $|\theta_{\omega A}(F)| = 3$. Then there exists such an $\omega \delta$ -fibered formation $H \in \theta_{\omega A}(F)$ that $(1) \not< H \in F$. Suppose that $G$ is a group of the minimal order in $F \setminus H$. Since $F$ and $H$ are formations then $G$ is a monolithic group with the monolith $R = G''$. Then there exists an $\omega \delta$ -fibered formation $\omega F(G, \delta) \not< \theta_{\omega A}(F)$. Note that $G \not< H$ and, so, $\omega F(G, \delta) \subseteq H$. Since $(1) \subseteq \omega F(G, \delta) \subseteq H$. $\omega F(G, \delta) \not< H$ and $|\theta_{\omega A}(F)| = 3$ then $\omega F(G, \delta) = (1)$ or $\omega F(G, \delta) = F$. If $\omega F(G, \delta) = (1)$ then $G \equiv 1$ and $G$ is a contradiction. Consequently, $\omega F(G, \delta) = F$.

Since $F$ is a $Q$-closed class of groups then $G/R \in F$ and $\omega F(G/R, \delta) \subseteq F$. Thus, $\omega F(G/R, \delta) \in \theta_{\omega A}(F)$. According to the definition of the $H$-coarcadic of the group it follows that $G/R \in H$. It follows that $(1) \subseteq \omega F(G, \delta) \not< H$. Since $\omega F(G, \delta) \in \theta_{\omega A}(F)$ and $|\theta_{\omega A}(F)| = 3$ then $\omega F(G, \delta) = (1)$ or $\omega F(G, \delta) = H$. If $\omega F(G, \delta) = (1)$ then $G \equiv R$ and $G$ is a simple group. Since $G \in F$ then $G$ is $\omega$-solvable group. In a view of $F = \omega F(G, \delta)$ and according to the Corollary 1, it follows that $|\theta_{\omega A}(F)| = 2$ which is a contradiction. Consequently, $\omega F(G/R, \delta) = H$ and, so, $\omega F(G/R, \delta)$ is the unique maximal $\omega \delta$ -fibered subformation of $F$. The theorem is proved.

**Conclusion:**

In the paper, we studied properties of the lattices of $\omega$-fibered formations of groups with the given...
direction \( \delta \). In this work the following tasks have been solved. We established properties of the lattice \( \theta_{\omega \delta}(F) \) of all \( \omega \)-fibered subformation with the direction \( \delta \) of the formation \( F \) in the case \( |\theta_{\omega \delta}(F)| = 2 \) (Theorem 1) and in the case \( |\theta_{\omega \delta}(F)| = 3 \) (Theorem 3). In the Theorem 2 we established conditions under which the lattice \( \theta_{\omega \delta}(F) \) of all \( \omega \)-fibered subformations of the given \( \omega \)-fibered formation \( F \) with the direction \( \delta \) has a finite number of elements not more 3. The further interest in the study of lattice properties of formations of finite groups has the lattices with the cardinality more or equal to four.

References